Spacing statistics of model spectra related to Farey sequences

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Introduction

It is well known [1] that quantum chaos theory has applications in classical wave physics, such as in microwave cavities and in acoustics, but its potential for providing new insights for characterizing plasma waves in complex geometries, such as the H-1NF stellarator at The Australian National University, has been little explored.

One signature of quantum chaos is the failure of standard semiclassical (EBK) quantization due to the occurrence of chaotic ray paths in the WKB approximation. Another is the appearance of level repulsion in the statistics of the separation between nearest-neighbour eigenvalues, the primary diagnostic [2] being the probability distribution function (PDF), \( P(s) \), of the energy level separations, \( s_i = E_{i+1} - E_i \), with the energy scaled so as to make the average separation unity. Experimentally, the presence of quantum chaos might be seen as a lack of spatial coherence in wave activity.


This suggests quantum chaos theory as a promising framework for characterizing the normal-mode spectra of waves and instabilities in stellarators. However a careful analysis of the spectrum of ideal-MHD interchange modes in a separable cylindrical approximation [5] revealed non-generic behaviour of the spectral statistics—a bimodal PDF, rather than the expected Poisson distribution. The non-genericity of this separable case indicates that caution must be applied in applying conventional quantum chaos theory in non-separable geometries.

The study [5] indicated that the non-generic behaviour of ideal-MHD interchange modes was due to the peculiar feature of the dispersion relation for these modes that the eigenvalues in the short-wavelength limit depend only on the direction of the wave vector, not on its magnitude. (This is unusual behaviour, but it is shared with internal gravity waves in geophysical fluid dynamics.) It was suggested in [5] that the detailed features of the spectrum could be understood from the properties of Farey sequences.

The Farey sequence \( \mathcal{F}(Q) \) is the set of all rational numbers \( p/q \) between 0 and 1, \( 1 < q \leq Q \), arranged in order of increasing arithmetic size and keeping only mutually prime pairs of integers \( p \) and \( q \). They are important in number theory [6] and have application in various dynamical systems problems, such as the theory of mode locking in circle maps [7, p. 391].
They even have a connection with the famous Riemann hypothesis [8].

**Toy eigenvalue problem**

To elucidate universal aspects of such problems we study the energy spectrum \{E_{n,m}\} for a “toy” quantum mechanical Hamiltonian \(H = p_\theta / p_\phi\) where the configuration space is the 2-torus \(\theta \in [0, 2\pi), \phi \in [0, 2\pi)\) with periodic boundary conditions. In the semiclassical approximation we see that \(H\) depends only on the direction of \(p\), not its magnitude, as for MHD interchange modes and internal gravity waves.

The eigenvalue problem is the time-independent Schrödinger equation, \(H\psi = E\psi\), the eigenfunctions being exp\([i(m\theta + n\phi)]/4\pi^2\), where \(m\) and \(n\) are integers. The eigenvalues \(E_{n,m} = n/m\) \((m \neq 0)\).

Note the singular nature of the spectrum—it is discrete, yet infinitely dense, the rationals being dense on the real line. Also, the spectrum is infinitely degenerate as eigenvalues are repeated whenever \(m\) and \(n\) have a common factor. Mathematically such a spectrum, neither point nor continuous, belongs to the essential spectrum [9].

In order to analyze this spectrum using standard quantum chaos techniques we first regularize it by bounding the region of the \(m,n\) lattice studied, and then allowing the bound to increase indefinitely. Fortunately the PDF \(P(s)\) is independent of the precise shape of the bounding line when we follow standard practice [2] in renormalizing (unfolding) the energy levels to make the average spacing unity. Thus we adopt the simplest choice, taking the bounded region to be the triangle \(0 \leq n \leq m \leq m_{\text{max}}\). As the points \((n,m)\) form a lattice in the plane with mean areal density of 1, we can estimate the asymptotic, large-\(m_{\text{max}}\) behaviour of the number of levels, \(N_{\text{max}}\), from the area of the bounding triangle: the \(m\) axis, the line \(n = m\) and the line \(m = 1\), which gives the “Weyl formula” [2] \(N_{\text{max}} \sim m_{\text{max}}^2/2\).

The list \(G(m_{\text{max}}) \equiv \{E_{n,m}\}\), sorted into a non-decreasing sequence \(\{E_i| i = 1,2,\ldots,N_{\text{max}}\}\) is very similar to the Farey sequence \(F(m_{\text{max}})\) except for the high degeneracy (multiplicity) of numerically identical levels, especially when \(n/m\) is a low-order rational.

Define the renormalized (unfolded) energy as \(E_{n,m} \equiv N_{\text{max}} E_{n,m}\). The normalization by \(N_{m_{\text{max}}}\) ensures that \(E_{N_{\text{max}}} = N_{\text{max}}\), so the mean slope of the Devil’s staircase shown in Fig. 1(a) is unity. The large vertical steps visible in Fig. 1(a) are due to the high degeneracy at low-order
Figure 2: Separation statistics: (a) for the model Hamiltonian; (b) for the Farey sequence. The solid curves are from the Farey spacing measure, Eq. (2). In (b) the short-dashed curve is for the Poisson Process of the generic integrable problem and the long-dashed curve is that for the Gaussian orthogonal ensemble of random matrices (quantum chaotic case).

rational values of $n/m$.

This high degeneracy is also the cause of the delta function spike at the origin visible in the level-separation probability distribution plot shown in Fig. 2(a).

**Farey statistics**

The tail in Fig. 2(a) is due to the non-degenerate component of the spectrum, obtained by reducing all fractions $p/q$ to lowest terms and deleting duplications. Thus the eigenvalues in this set are $N_Q$ times the terms of the Farey sequence $\mathcal{F}(Q)$.

To study the statistics of this non-degenerate component it is natural to define the *Farey spectrum* $\{E_i^F\}$ as $N^F(Q)$ times the terms of the Farey sequence $\mathcal{F}(Q)$, where $N^F(Q)$ is the number of terms in $\mathcal{F}(Q)$. The asymptotic behaviour of $N^F(Q)$ in the large-$n$ limit is given [7, p. 391] by $N^F(Q) \sim 3Q^2/\pi^2 + O(Q \ln Q)$. The staircase plot and separation distribution $P^F(s)$ for the Farey spectrum are given in Fig. 1(b) and Fig. 2(b), respectively.

It is a standard result in the theory of Farey sequences [6, p. 301] that the smallest and largest nearest-neighbour spacings in $\mathcal{F}(Q)$ are given respectively by

$$\frac{1}{Q(Q-1)} \quad \text{and} \quad \frac{1}{Q},$$

so that the support of the tail component of $P(s)$ in Fig. 2(a) becomes $[1/2, \infty)$ in the limit $Q \to \infty$, while that of $P^F(s)$ in Fig. 2(b) is $[3/\pi^2, \infty)$.

Augustin et al. [8], eq. (1.9), derive the spacing density for the Farey sequence as

$$g_1(t) = \begin{cases} 0, & \text{for } 0 \leq t \leq \frac{3}{\pi^2}, \\ \frac{6}{\pi^2} \ln \left( \frac{\pi^2 t}{3} \right), & \text{for } \frac{3}{\pi^2} \leq t \leq \frac{12}{\pi^2}, \\ \frac{12}{\pi^2} \ln \left[ \frac{\pi^2 t}{6} \left( 1 - \sqrt{1 - \frac{12}{\pi^2} t} \right) \right], & \text{for } \frac{12}{\pi^2} \leq t. \end{cases}$$

The solid curve in Fig. 2(b) is obtained by setting $P^F(s) = g_1(s)$ and is seen to agree well with the numerical results.
The solid curve in Fig. 2(a) is obtained by setting $P(s) = \left[ \frac{N^F(Q)}{N(Q)} \right]^2 g_1 (\frac{N^F s}{N})$ and agrees well with the tail of the histogram. The ratio of the area of the tail in Fig. 2(a) to the strength of the delta function Fig. 2(a) is $\frac{N^F(Q)}{[N(Q) - N^F(Q)]} \approx 1.55$.

We have verified that the probability distributions remain unchanged if subrange of the spectra are used, in agreement with the result included in Theorem 1.1 of [8] that the convergence to a probability measure is independent of the interval chosen.

**Conclusion**

The robustness of the statistics to the choice of subrange of the spectra is a form of universality. However, it was shown in [5] that there are $O(1/m^2)$ corrections to the spectrum of ideal magnetohydrodynamic interchange modes and that these affect the spacing distribution because of the $O(m^2_{\text{max}})$ normalization factor in the spectrum. This suggests that one should study the new one-parameter family of spectra $\mathcal{E}_{n,m} \equiv n/m + \epsilon/m^2$ in order to understand this physical application. In the present paper we have looked at the case $\epsilon = 0$. Non-zero $\epsilon$ would break the strong degeneracy that causes the delta function seen in Fig. 2(a), broadening it into a peak of finite width at the origin, explaining the bimodal distribution seen in [5]. Finite-Larmor-radius effects [10] also need to be examined to make the study more physically relevant to plasma waves and instabilities.

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